3.2 Euler-Gleichungen

Lösungen

Aufgabe 1

Mit

$$\boldsymbol{v} = v_x \boldsymbol{e_x} + v_y \boldsymbol{e_y} + v_z \boldsymbol{e_z}$$

und

$$\nabla = \boldsymbol{e_x} \frac{\partial}{\partial x} + \boldsymbol{e_y} \frac{\partial}{\partial y} + \boldsymbol{e_z} \frac{\partial}{\partial z}$$

gilt

$$\boldsymbol{v} \cdot \nabla = v_x \frac{\partial}{\partial x} + v_y \frac{\partial}{\partial y} + v_z \frac{\partial}{\partial z} \quad .$$

Damit folgt:

$$\begin{aligned}(\boldsymbol{v} \cdot \nabla) \boldsymbol{v} &= v_x \frac{\partial \boldsymbol{v}}{\partial x} + v_y \frac{\partial \boldsymbol{v}}{\partial y} + v_z \frac{\partial \boldsymbol{v}}{\partial z} \\ &= \left(v_x \frac{\partial v_x}{\partial x} + v_y \frac{\partial v_x}{\partial y} + v_z \frac{\partial v_x}{\partial z} \right) \boldsymbol{e_x} + \left(v_x \frac{\partial v_y}{\partial x} + v_y \frac{\partial v_y}{\partial y} + v_z \frac{\partial v_y}{\partial z} \right) \boldsymbol{e_y} \\ &\quad + \left(v_x \frac{\partial v_z}{\partial x} + v_y \frac{\partial v_z}{\partial y} + v_z \frac{\partial v_z}{\partial z} \right) \boldsymbol{e_z}\end{aligned}$$

Damit lautet der Impulssatz:

$$\begin{aligned}\frac{\partial v_x}{\partial t} + v_x \frac{\partial v_x}{\partial x} + v_y \frac{\partial v_x}{\partial y} + v_z \frac{\partial v_x}{\partial z} &= -\frac{1}{\rho_0} \frac{\partial p}{\partial x} \\ \frac{\partial v_y}{\partial t} + v_x \frac{\partial v_y}{\partial x} + v_y \frac{\partial v_y}{\partial y} + v_z \frac{\partial v_y}{\partial z} &= -\frac{1}{\rho_0} \frac{\partial p}{\partial y} \\ \frac{\partial v_z}{\partial t} + v_x \frac{\partial v_z}{\partial x} + v_y \frac{\partial v_z}{\partial y} + v_z \frac{\partial v_z}{\partial z} &= -\frac{1}{\rho_0} \frac{\partial p}{\partial z}\end{aligned}$$

Für die Kontinuitätsgleichung ergibt sich

$$\nabla \cdot \boldsymbol{v} = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} = 0 \quad .$$

Aufgabe 2

Nach Aufgabe 1 gilt:

$$2(\boldsymbol{v}\cdot\nabla)\boldsymbol{v}=2\left(v_x\frac{\partial v_x}{\partial x}+v_y\frac{\partial v_x}{\partial y}+v_z\frac{\partial v_x}{\partial z}\right)\boldsymbol{e_x}+2\left(v_x\frac{\partial v_y}{\partial x}+v_y\frac{\partial v_y}{\partial y}+v_z\frac{\partial v_y}{\partial z}\right)\boldsymbol{e_y}$$
$$+2\left(v_x\frac{\partial v_z}{\partial x}+v_y\frac{\partial v_z}{\partial y}+v_z\frac{\partial v_z}{\partial z}\right)\boldsymbol{e_z}$$

Die rechte Seite lässt sich weiter umformen zu

$$2(\boldsymbol{v}\cdot\nabla)\boldsymbol{v}=\left(\boldsymbol{e_x}\frac{\partial}{\partial x}+\boldsymbol{e_y}\frac{\partial}{\partial y}+\boldsymbol{e_z}\frac{\partial}{\partial z}\right)\left(v_x^2+v_y^2+v_z^2\right)$$
$$-\frac{\partial}{\partial x}\left(v_y^2+v_z^2\right)\boldsymbol{e_x}-\frac{\partial}{\partial y}\left(v_x^2+v_z^2\right)\boldsymbol{e_y}-\frac{\partial}{\partial z}\left(v_x^2+v_y^2\right)\boldsymbol{e_z}$$
$$+2\left(v_y\frac{\partial v_x}{\partial y}+v_z\frac{\partial v_x}{\partial z}\right)\boldsymbol{e_x}+2\left(v_x\frac{\partial v_y}{\partial x}+v_z\frac{\partial v_y}{\partial z}\right)\boldsymbol{e_y}+2\left(v_x\frac{\partial v_z}{\partial x}+v_y\frac{\partial v_z}{\partial y}\right)\boldsymbol{e_z}.$$

Für den ersten Term auf der rechten Seite gilt:

$$\left(\boldsymbol{e_x}\frac{\partial}{\partial x}+\boldsymbol{e_y}\frac{\partial}{\partial y}+\boldsymbol{e_z}\frac{\partial}{\partial z}\right)\left(v_x^2+v_y^2+v_z^2\right)=\nabla(\boldsymbol{v}\cdot\boldsymbol{v})$$

Zusammengefasst nach Komponenten folgt für die übrigen Terme:

$$\left(2v_y\frac{\partial v_x}{\partial y}+2v_z\frac{\partial v_x}{\partial z}-\frac{\partial}{\partial x}\left(v_y^2+v_z^2\right)\right)\boldsymbol{e_x}=2\left(v_y\left(\frac{\partial v_x}{\partial y}-\frac{\partial v_y}{\partial x}\right)+v_z\left(\frac{\partial v_x}{\partial z}-\frac{\partial v_z}{\partial x}\right)\right)\boldsymbol{e_x}$$
$$=2v_y\boldsymbol{e_y}\times\left(\frac{\partial v_x}{\partial y}-\frac{\partial v_y}{\partial x}\right)\boldsymbol{e_z}-2v_z\boldsymbol{e_z}\times\left(\frac{\partial v_x}{\partial z}-\frac{\partial v_z}{\partial x}\right)\boldsymbol{e_y}$$

$$\left(2v_x\frac{\partial v_y}{\partial x}+2v_z\frac{\partial v_y}{\partial z}-\frac{\partial}{\partial y}\left(v_x^2+v_z^2\right)\right)\boldsymbol{e_y}=2\left(v_x\left(\frac{\partial v_y}{\partial x}-\frac{\partial v_x}{\partial y}\right)+v_z\left(\frac{\partial v_y}{\partial z}-\frac{\partial v_z}{\partial y}\right)\right)\boldsymbol{e_y}$$
$$=-2v_x\boldsymbol{e_x}\times\left(\frac{\partial v_y}{\partial x}-\frac{\partial v_x}{\partial y}\right)\boldsymbol{e_z}+2v_z\boldsymbol{e_z}\times\left(\frac{\partial v_y}{\partial z}-\frac{\partial v_z}{\partial y}\right)\boldsymbol{e_x}$$

$$\left(2v_x\frac{\partial v_z}{\partial x}+2v_y\frac{\partial v_z}{\partial y}-\frac{\partial}{\partial z}\left(v_x^2+v_y^2\right)\right)\boldsymbol{e_z}=2\left(v_x\left(\frac{\partial v_z}{\partial x}-\frac{\partial v_x}{\partial z}\right)+v_y\left(\frac{\partial v_z}{\partial y}-\frac{\partial v_y}{\partial z}\right)\right)\boldsymbol{e_z}$$
$$=2v_x\boldsymbol{e_x}\times\left(\frac{\partial v_z}{\partial x}-\frac{\partial v_x}{\partial z}\right)\boldsymbol{e_y}-2v_y\boldsymbol{e_y}\times\left(\frac{\partial v_z}{\partial y}-\frac{\partial v_y}{\partial z}\right)\boldsymbol{e_x}$$

Addition der rechten Seiten ergibt

$$2v_x\boldsymbol{e_x}\times\left(\left(\frac{\partial v_z}{\partial x}-\frac{\partial v_x}{\partial z}\right)\boldsymbol{e_y}+\left(\frac{\partial v_x}{\partial y}-\frac{\partial v_y}{\partial x}\right)\boldsymbol{e_z}\right)$$
$$+2v_y\boldsymbol{e_y}\times\left(\left(\frac{\partial v_y}{\partial z}-\frac{\partial v_z}{\partial y}\right)\boldsymbol{e_x}+\left(\frac{\partial v_x}{\partial y}-\frac{\partial v_y}{\partial x}\right)\boldsymbol{e_z}\right)$$
$$+2v_z\boldsymbol{e_z}\times\left(\left(\frac{\partial v_y}{\partial z}-\frac{\partial v_z}{\partial y}\right)\boldsymbol{e_x}+\left(\frac{\partial v_z}{\partial x}-\frac{\partial v_x}{\partial z}\right)\boldsymbol{e_y}\right)=-2\boldsymbol{v}\times\boldsymbol{\omega}$$

mit

$$\begin{aligned}\boldsymbol{\omega}=\nabla\times\boldsymbol{v}&=\left(\boldsymbol{e}_x\frac{\partial}{\partial x}+\boldsymbol{e}_y\frac{\partial}{\partial y}+\boldsymbol{e}_z\frac{\partial}{\partial z}\right)\times\left(v_x\boldsymbol{e}_x+v_y\boldsymbol{e}_y+v_z\boldsymbol{e}_z\right)\\&=\frac{\partial v_y}{\partial x}\boldsymbol{e}_z-\frac{\partial v_z}{\partial x}\boldsymbol{e}_y-\frac{\partial v_x}{\partial y}\boldsymbol{e}_z+\frac{\partial v_z}{\partial y}\boldsymbol{e}_x+\frac{\partial v_x}{\partial z}\boldsymbol{e}_y-\frac{\partial v_y}{\partial z}\boldsymbol{e}_x\\&=\left(\frac{\partial v_z}{\partial y}-\frac{\partial v_y}{\partial z}\right)\boldsymbol{e}_x+\left(\frac{\partial v_x}{\partial z}-\frac{\partial v_z}{\partial x}\right)\boldsymbol{e}_y+\left(\frac{\partial v_y}{\partial x}-\frac{\partial v_x}{\partial y}\right)\boldsymbol{e}_z\ .\end{aligned}$$

Damit ist gezeigt:

$$2\left(\boldsymbol{v}\cdot\nabla\right)\boldsymbol{v}=\nabla\left(\boldsymbol{v}\cdot\boldsymbol{v}\right)-2\,\boldsymbol{v}\times\left(\nabla\times\boldsymbol{v}\right)$$