3.4 Wirbelsätze

Lösungen

Aufgabe 1

Mit

$$\nabla\times\boldsymbol{v}=\left(\boldsymbol{e}_x\frac{\partial}{\partial x}+\boldsymbol{e}_y\frac{\partial}{\partial y}+\boldsymbol{e}_z\frac{\partial}{\partial z}\right)\times\left(v_x\boldsymbol{e}_x+v_y\boldsymbol{e}_y+v_z\boldsymbol{e}_z\right)$$
$$=\left(\frac{\partial v_z}{\partial y}-\frac{\partial v_y}{\partial z}\right)\boldsymbol{e}_x+\left(\frac{\partial v_x}{\partial z}-\frac{\partial v_z}{\partial x}\right)\boldsymbol{e}_y+\left(\frac{\partial v_y}{\partial x}-\frac{\partial v_x}{\partial y}\right)\boldsymbol{e}_z$$

folgt

$$\nabla\cdot\left(\nabla\times\boldsymbol{v}\right)=\left(\boldsymbol{e}_x\frac{\partial}{\partial x}+\boldsymbol{e}_y\frac{\partial}{\partial y}+\boldsymbol{e}_z\frac{\partial}{\partial z}\right)$$
$$\cdot\left(\left(\frac{\partial v_z}{\partial y}-\frac{\partial v_y}{\partial z}\right)\boldsymbol{e}_x+\left(\frac{\partial v_x}{\partial z}-\frac{\partial v_z}{\partial x}\right)\boldsymbol{e}_y+\left(\frac{\partial v_y}{\partial x}-\frac{\partial v_x}{\partial y}\right)\boldsymbol{e}_z\right)$$
$$=\frac{\partial^2 v_z}{\partial x\partial y}-\frac{\partial^2 v_y}{\partial x\partial z}+\frac{\partial^2 v_x}{\partial z\partial y}-\frac{\partial^2 v_z}{\partial x\partial y}+\frac{\partial^2 v_y}{\partial z\partial x}-\frac{\partial^2 v_x}{\partial y\partial z}=0 \quad .$$

Aufgabe 2

Mit

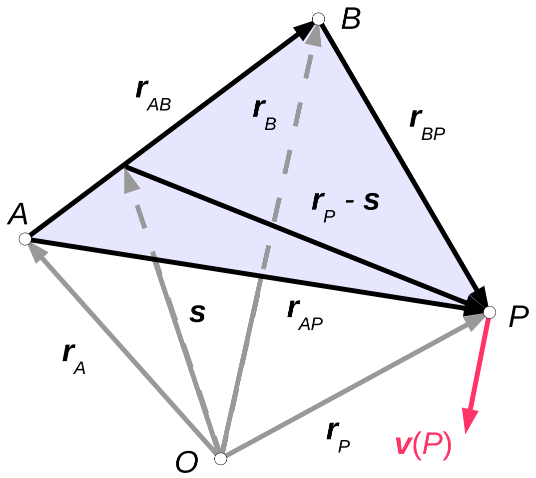
$$\nabla\phi=\frac{\partial\phi}{\partial x}\boldsymbol{e}_x+\frac{\partial\phi}{\partial y}\boldsymbol{e}_y+\frac{\partial\phi}{\partial z}\boldsymbol{e}_z$$

gilt:

$$\nabla\times\left(\nabla\phi\right)=\left(\frac{\partial}{\partial x}\boldsymbol{e}_x+\frac{\partial}{\partial y}\boldsymbol{e}_y+\frac{\partial}{\partial z}\boldsymbol{e}_z\right)\times\left(\frac{\partial\phi}{\partial x}\boldsymbol{e}_x+\frac{\partial\phi}{\partial y}\boldsymbol{e}_y+\frac{\partial\phi}{\partial z}\boldsymbol{e}_z\right)$$
$$=\frac{\partial^2\phi}{\partial x\partial y}\boldsymbol{e}_x\times\boldsymbol{e}_y+\frac{\partial^2\phi}{\partial x\partial z}\boldsymbol{e}_x\times\boldsymbol{e}_z+\frac{\partial^2\phi}{\partial y\partial x}\boldsymbol{e}_y\times\boldsymbol{e}_x+\frac{\partial^2\phi}{\partial y\partial z}\boldsymbol{e}_y\times\boldsymbol{e}_z$$
$$+\frac{\partial^2\phi}{\partial z\partial x}\boldsymbol{e}_z\times\boldsymbol{e}_x+\frac{\partial^2\phi}{\partial z\partial y}\boldsymbol{e}_z\times\boldsymbol{e}_y$$
$$=\left(\frac{\partial^2\phi}{\partial x\partial y}-\frac{\partial^2\phi}{\partial y\partial x}\right)\boldsymbol{e}_x\times\boldsymbol{e}_y+\left(\frac{\partial^2\phi}{\partial x\partial z}-\frac{\partial^2\phi}{\partial z\partial x}\right)\boldsymbol{e}_x\times\boldsymbol{e}_z$$
$$+\left(\frac{\partial^2\phi}{\partial y\partial z}-\frac{\partial^2\phi}{\partial z\partial y}\right)\boldsymbol{e}_y\times\boldsymbol{e}_z=\boldsymbol{0}$$

Aufgabe 3

a) <u>Endlicher gerader Abschnitt</u>



Das Gesetz von Biot-Savart lautet:

$$\boldsymbol{v}(P) = -\frac{\Gamma}{4\pi}\int_A^B \frac{(\boldsymbol{r}_P - \boldsymbol{s}) \times d\boldsymbol{s}}{|\boldsymbol{r}_P - \boldsymbol{s}|^3}$$

Die Abbildung zeigt:

$$\boldsymbol{s} = \boldsymbol{r}_A + \lambda \boldsymbol{r}_{AB}, \quad 0 \le \lambda \le 1$$

$$\boldsymbol{r}_{AB} = \boldsymbol{r}_B - \boldsymbol{r}_A$$

$$\boldsymbol{r}_P - \boldsymbol{s} = \boldsymbol{r}_P - \boldsymbol{r}_A - \lambda \boldsymbol{r}_{AB} = \boldsymbol{r}_{AP} - \lambda \boldsymbol{r}_{AB}$$

Daraus folgt:

$$d\boldsymbol{s} = \boldsymbol{r}_{AB}\, d\lambda$$

$$(\boldsymbol{r}_P - \boldsymbol{s}) \times d\boldsymbol{s} = (\boldsymbol{r}_{AP} - \lambda \boldsymbol{r}_{AB}) \times \boldsymbol{r}_{AB}\, d\lambda = \boldsymbol{r}_{AP} \times \boldsymbol{r}_{AB}\, d\lambda = -\boldsymbol{r}_{AB} \times \boldsymbol{r}_{AP}\, d\lambda$$

$$|\boldsymbol{r}_P - \boldsymbol{s}|^2 = \boldsymbol{r}_{AP} \cdot \boldsymbol{r}_{AP} - 2\boldsymbol{r}_{AP} \cdot \boldsymbol{r}_{AB}\lambda + \boldsymbol{r}_{AB} \cdot \boldsymbol{r}_{AB}\lambda^2$$

Mit

$$c = \boldsymbol{r}_{AP} \cdot \boldsymbol{r}_{AP} = |\boldsymbol{r}_{AP}|^2, \quad b = -2\boldsymbol{r}_{AP}\boldsymbol{r}_{AB}, \quad a = \boldsymbol{r}_{AB} \cdot \boldsymbol{r}_{AB} = |\boldsymbol{r}_{AB}|^2$$

gilt:

$$|\boldsymbol{r}_P - \boldsymbol{s}| = \sqrt{a\lambda^2 + b\lambda + c}$$

Damit folgt:

$$\boldsymbol{v}(P) = \frac{\Gamma}{4\pi}(\boldsymbol{r}_{AB} \times \boldsymbol{r}_{AP})\int_0^1 \frac{d\lambda}{(a\lambda^2 + b\lambda + c)^{3/2}}$$

Mit $\Delta = 4ac - b^2$ berechnet sich das Integral zu

$$\int_0^1 \frac{d\lambda}{(a\lambda^2 + b\lambda + c)^{3/2}} = \left[\frac{4a\lambda + 2b}{\Delta\sqrt{a\lambda^2 + b\lambda + c}}\right]_{\lambda=0}^{\lambda=1} = \frac{1}{\Delta}\left(\frac{4a + 2b}{\sqrt{a + b + c}} - \frac{2b}{\sqrt{c}}\right) \quad .$$

Mit der Lagrangeschen Identität

$$(\boldsymbol{a} \times \boldsymbol{b}) \cdot (\boldsymbol{c} \times \boldsymbol{d}) = (\boldsymbol{a} \cdot \boldsymbol{c})(\boldsymbol{b} \cdot \boldsymbol{d}) - (\boldsymbol{b} \cdot \boldsymbol{c})(\boldsymbol{a} \cdot \boldsymbol{d})$$

folgt

$$\Delta = 4\left((\boldsymbol{r}_{AB} \cdot \boldsymbol{r}_{AB})(\boldsymbol{r}_{AP} \cdot \boldsymbol{r}_{AP}) - (\boldsymbol{r}_{AP} \cdot \boldsymbol{r}_{AB})^2\right) = 4(\boldsymbol{r}_{AB} \times \boldsymbol{r}_{AP}) \cdot (\boldsymbol{r}_{AB} \times \boldsymbol{r}_{AP})$$
$$= 4|\boldsymbol{r}_{AB} \times \boldsymbol{r}_{AP}|^2 \quad .$$

Die übrigen Terme berechnen sich zu

$$4a+2b=4\left(\boldsymbol{r}_{AB}\cdot\boldsymbol{r}_{AB}-\boldsymbol{r}_{AP}\cdot\boldsymbol{r}_{AB}\right)=4\,\boldsymbol{r}_{AB}\left(\boldsymbol{r}_{AB}-\boldsymbol{r}_{AP}\right)=-4\,\boldsymbol{r}_{AB}\cdot\boldsymbol{r}_{BP}\ ,$$

$$a+b+c=\boldsymbol{r}_{AB}\cdot\boldsymbol{r}_{AB}-2\,\boldsymbol{r}_{AP}\cdot\boldsymbol{r}_{AB}+\boldsymbol{r}_{AP}\cdot\boldsymbol{r}_{AP}=\left(\boldsymbol{r}_{AB}-\boldsymbol{r}_{AP}\right)\cdot\left(\boldsymbol{r}_{AB}-\boldsymbol{r}_{AP}\right)$$
$$=\left|\boldsymbol{r}_{AB}-\boldsymbol{r}_{AP}\right|^2=\left|\boldsymbol{r}_{BP}\right|^2$$

und $\dfrac{2b}{\sqrt{c}}=-4\dfrac{\boldsymbol{r}_{AP}\cdot\boldsymbol{r}_{AB}}{\left|\boldsymbol{r}_{AP}\right|}$.

Einsetzen in das Gesetz von Biot-Savart ergibt:

$$\boldsymbol{v}(P)=\frac{\Gamma}{4\pi}\frac{\boldsymbol{r}_{AB}\times\boldsymbol{r}_{AP}}{4\left|\boldsymbol{r}_{AB}\times\boldsymbol{r}_{AP}\right|^2}\left(-4\frac{\boldsymbol{r}_{AB}\cdot\boldsymbol{r}_{BP}}{\left|\boldsymbol{r}_{BP}\right|}+4\frac{\boldsymbol{r}_{AP}\cdot\boldsymbol{r}_{AB}}{\left|\boldsymbol{r}_{AP}\right|}\right)$$
$$=\frac{\Gamma}{4\pi}\frac{\boldsymbol{r}_{AB}\times\boldsymbol{r}_{AP}}{\left|\boldsymbol{r}_{AB}\times\boldsymbol{r}_{AP}\right|^2}\left(\frac{\boldsymbol{r}_{AB}\cdot\boldsymbol{r}_{AP}}{\left|\boldsymbol{r}_{AP}\right|}-\frac{\boldsymbol{r}_{AB}\cdot\boldsymbol{r}_{BP}}{\left|\boldsymbol{r}_{BP}\right|}\right)$$

Wegen

$$\boldsymbol{r}_{AB}\times\boldsymbol{r}_{AP}=\left(\boldsymbol{r}_{AP}-\boldsymbol{r}_{BP}\right)\times\boldsymbol{r}_{AP}=-\boldsymbol{r}_{BP}\times\boldsymbol{r}_{AP}=\boldsymbol{r}_{AP}\times\boldsymbol{r}_{BP}$$

gilt auch:

$$\boldsymbol{v}(P)=\frac{\Gamma}{4\pi}\frac{\boldsymbol{r}_{AP}\times\boldsymbol{r}_{BP}}{\left|\boldsymbol{r}_{AP}\times\boldsymbol{r}_{BP}\right|^2}\left(\frac{\boldsymbol{r}_{AB}\cdot\boldsymbol{r}_{AP}}{\left|\boldsymbol{r}_{AP}\right|}-\frac{\boldsymbol{r}_{AB}\cdot\boldsymbol{r}_{BP}}{\left|\boldsymbol{r}_{BP}\right|}\right)$$

b) Halbunendlicher gerader Abschnitt

Der Beitrag einer im Punkt *A* beginnenden, durch *B* verlaufenden Halbgeraden ergibt sich als Grenzwert $\lambda\to\infty$ aus dem Ergebnis für die Strecke *AB*. Mit

$$\int_0^\infty\frac{d\lambda}{\left(a\lambda^2+b\lambda+c\right)^{3/2}}=\left[\frac{4a\lambda+2b}{\Delta\sqrt{a\lambda^2+b\lambda+c}}\right]_{\lambda=0}^{\lambda\to\infty}=\frac{1}{\Delta}\left(\frac{4a}{\sqrt{a}}-\frac{2b}{\sqrt{c}}\right)$$
$$=\frac{1}{\left|\boldsymbol{r}_{AB}\times\boldsymbol{r}_{AP}\right|^2}\left(\left|\boldsymbol{r}_{AB}\right|+\frac{\boldsymbol{r}_{AP}\cdot\boldsymbol{r}_{AB}}{\left|\boldsymbol{r}_{AP}\right|}\right)$$
$$=\frac{1/\left|\boldsymbol{r}_{AB}\right|}{\left|\boldsymbol{r}_{AB}/\left|\boldsymbol{r}_{AB}\right|\times\boldsymbol{r}_{AP}\right|^2}\left(1+\frac{\boldsymbol{r}_{AB}\cdot\boldsymbol{r}_{AP}}{\left|\boldsymbol{r}_{AB}\right|\left|\boldsymbol{r}_{AP}\right|}\right)$$

folgt:

$$\boldsymbol{v}(P)=\frac{\Gamma}{4\pi}\frac{\boldsymbol{r}_{AB}/\left|\boldsymbol{r}_{AB}\right|\times\boldsymbol{r}_{AP}}{\left|\boldsymbol{r}_{AB}/\left|\boldsymbol{r}_{AB}\right|\times\boldsymbol{r}_{AP}\right|^2}\left(1+\frac{\boldsymbol{r}_{AB}}{\left|\boldsymbol{r}_{AB}\right|}\cdot\frac{\boldsymbol{r}_{AP}}{\left|\boldsymbol{r}_{AP}\right|}\right)$$

c) Unendlicher Wirbelfaden

In diesem Fall geht die untere Integrationsgrenze gegen minus unendlich und die obere gegen unendlich. Mit

$$\int_{-\infty}^{\infty} \frac{d\lambda}{\left(a\lambda^2+b\lambda+c\right)^{3/2}} = \left[\frac{4a\lambda+2b}{\Delta\sqrt{a\lambda^2+b\lambda+c}}\right]_{\lambda\to-\infty}^{\lambda\to\infty} = \frac{1}{\Delta}\left(\frac{4a}{\sqrt{a}}+\frac{4a}{\sqrt{a}}\right) = \frac{8\sqrt{a}}{\Delta}$$

$$= \frac{2|\boldsymbol{r}_{AB}|}{|\boldsymbol{r}_{AB}\times\boldsymbol{r}_{AP}|^2} = \frac{2}{|\boldsymbol{r}_{AB}|}\frac{1}{\left|\boldsymbol{r}_{AB}/|\boldsymbol{r}_{AB}|\times\boldsymbol{r}_{AP}\right|^2}$$

folgt:

$$\boldsymbol{v}(P) = \frac{\Gamma}{2\pi}\frac{\boldsymbol{r}_{AB}/|\boldsymbol{r}_{AB}|\times\boldsymbol{r}_{AP}}{\left|\boldsymbol{r}_{AB}/|\boldsymbol{r}_{AB}|\times\boldsymbol{r}_{AP}\right|^2}$$

d) Unendlicher Wirbelfaden entlang der *y*-Achse

Für einen Wirbelfaden entlang der *y*-Achse gilt: $\frac{\boldsymbol{r}_{AB}}{|\boldsymbol{r}_{AB}|} = \boldsymbol{e}_y$

Wird für Punkt *A* der Ursprung des Koordinatensystems gewählt, so gilt

$$\boldsymbol{r}_{AP} = \boldsymbol{r}_P = x\,\boldsymbol{e}_x + y\,\boldsymbol{e}_y + z\,\boldsymbol{e}_z$$

und

$$\frac{\boldsymbol{r}_{AB}}{|\boldsymbol{r}_{AB}|}\times\boldsymbol{r}_{AP} = \boldsymbol{e}_y\times\left(x\,\boldsymbol{e}_x + y\,\boldsymbol{e}_y + z\,\boldsymbol{e}_z\right) = z\,\boldsymbol{e}_x - x\,\boldsymbol{e}_z \quad .$$

Für die Geschwindigkeit folgt:

$$\boldsymbol{v}(P) = \frac{\Gamma}{2\pi}\frac{z\,\boldsymbol{e}_x - x\,\boldsymbol{e}_z}{x^2+z^2}$$