On the Zeros of the Fourier Transforms of Some Functions of Finite Duration

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1 Introduction

A function of finite duration is a function of the form

$$x(t) = \begin{cases} 0 & t < 0 \\ \phi(t/T) & 0 \le t \le T \\ 0 & t > T \end{cases} \tag{1}$$

where T is the duration of the function and ϕ is an arbitrary function. Functions of this type occur e.g. in structural dynamics where they describe impact loads. In a frequency or transient response analysis, it is necessary to determine a cut-off frequency, i.e. a frequency above which the magnitude of the Fourier transform is negligible. In many cases, the magnitude of the Fourier transform has zeros that are uniformly distributed, and the maximum of the magnitude between the zeros rapidly decreases with increasing frequency. Therefore, the cut-off frequency can be easily determined if the zeros are known.

This article shows that in some cases the Fourier transform can be expanded into an infinite series depending on the values of the derivatives of function $\phi(t/T)$ at $t = 0$ and $t = T$. This is possible if the derivatives exist and the series converges. Two special cases are identified where the zeros can be easily determined from the series and are found to be uniformly distributed.

2 Series Expansion of the Fourier Transform

The Fourier transform of a function of finite duration T reads

$$X(f) = \int_{-\infty}^{\infty} x(t) e^{-2\pi i f t} dt = \int_0^T \phi(t/T) e^{-2\pi i f t} dt. \tag{2}$$

By partial integration,

$$\int_0^T \frac{d^n \phi}{dt^n} e^{-2\pi i f t} dt = -\frac{1}{2\pi i f} \left(\left[\frac{d^n \phi}{dt^n} e^{-2\pi i f t} \right]_{t=0}^{t=T} - \int_0^T \frac{d^{n+1} \phi}{dt^{n+1}} e^{-2\pi i f t} dt \right). \tag{3}$$

Repeated application of Eq. 3 to Eq. 2 leads to

$$X(f) = -\frac{1}{2\pi i f}\sum_{n=0}^{\infty}\left(\frac{1}{2\pi i f}\right)^n \left[\frac{d^n\phi}{dt^n}e^{-2\pi i f t}\right]_{t=0}^{t=T} \tag{4}$$

or

$$X(fT) = \frac{iT}{2\pi fT}\sum_{n=0}^{\infty}\left(\frac{-i}{2\pi fT}\right)^n \left(\frac{d^n\phi}{d(t/T)^n}(1)e^{-2\pi i fT} - \frac{d^n\phi}{d(t/T)^n}(0)\right). \tag{5}$$

Eq. 5 shows that, if the series converges, the Fourier transform depends on fT and on the values of $\phi(t/T)$ and all its derivatives at $t = 0$ and $t = T$, provided these derivatives exist. Under the same conditions, the Fourier transform $X(fT)$ can be seen to be proportional to the duration T.

3 Zeros of the Fourier Transform

In the following two cases, the zeros of the Fourier transform can be easily determined from Eq. 5.

Case 1

$$\frac{d^n\phi}{d(t/T)^n}(0) = \frac{d^n\phi}{d(t/T)^n}(1) \quad \forall n \in \mathbb{N}_0 \tag{6}$$

With Eq. 6 Eq. 5 reads

$$X(fT) = \frac{iT}{2\pi fT}\left(e^{-2\pi i fT} - 1\right)\sum_{n=0}^{\infty}\left(\frac{-i}{2\pi fT}\right)^n \frac{d^n\phi}{d(t/T)^n}(1). \tag{7}$$

Eq. 7 shows that $X(fT) = 0$ for

$$f_nT = n, \quad n \in \mathbb{N}, \tag{8}$$

provided the infinite series converges for these values. In many practical cases, the series converges for $n > 1$.

Case 2

$$\frac{d^n\phi}{d(t/T)^n}(0) = -\frac{d^n\phi}{d(t/T)^n}(1) \quad \forall n \in \mathbb{N}_0 \tag{9}$$

With Eq. 9 Eq. 5 reads

$$X(fT) = \frac{iT}{2\pi fT}\left(e^{-2\pi i fT} + 1\right)\sum_{n=0}^{\infty}\left(\frac{-i}{2\pi fT}\right)^n \frac{d^n\phi}{d(t/T)^n}(1). \tag{10}$$

Now, $X(fT) = 0$ for

$$f_nT = \frac{2n+1}{2}, \quad n \in \mathbb{N}_0, \tag{11}$$

provided the infinite series converges.

4 Examples

Example 1

A function frequently occurring in structural dynamics or aeroelasticity is

$$\phi\left(\frac{t}{T}\right) = \sin^2\left(\pi\frac{t}{T}\right). \tag{12}$$

The derivatives are

$$\begin{aligned}
\frac{d\phi}{d(t/T)} &= 2\pi\sin\left(\pi\frac{t}{T}\right)\cos\left(\pi\frac{t}{T}\right) = \pi\sin\left(2\pi\frac{t}{T}\right)\\
\frac{d^2\phi}{d(t/T)^2} &= 2\pi^2\cos\left(2\pi\frac{t}{T}\right)\\
\frac{d^3\phi}{d(t/T)^3} &= -4\pi^3\sin\left(2\pi\frac{t}{T}\right)\\
\frac{d^4\phi}{d(t/T)^4} &= -8\pi^4\cos\left(2\pi\frac{t}{T}\right)\\
\frac{d^5\phi}{d(t/T)^5} &= 16\pi^5\sin\left(2\pi\frac{t}{T}\right)\\
&\cdots
\end{aligned}$$

The values at $t = 0$ and $t = T$ are

$$\begin{aligned}
\phi(0) &= 0, & \phi(1) &= 0\\
\frac{d\phi}{d(t/T)}(0) &= 0, & \frac{d\phi}{d(t/T)}(1) &= 0\\
\frac{d^2\phi}{d(t/T)^2}(0) &= 2\pi^2 = \frac{1}{2}(2\pi)^2, & \frac{d^2\phi}{d(t/T)^2}(1) &= 2\pi^2 = \frac{1}{2}(2\pi)^2\\
\frac{d^3\phi}{d(t/T)^3}(0) &= 0, & \frac{d^3\phi}{d(t/T)^3}(1) &= 0\\
\frac{d^4\phi}{d(t/T)^4}(0) &= -8\pi^4 = -\frac{1}{2}(2\pi)^4, & \frac{d^4\phi}{d(t/T)^4}(1) &= -8\pi^4 = -\frac{1}{2}(2\pi)^4\\
&\cdots & &\cdots
\end{aligned}$$

or

$$\frac{d^{2n}\phi}{d(t/T)^{2n}}(0) = \frac{d^{2n}\phi}{d(t/T)^{2n}}(T) = -\frac{(-1)^n}{2}(2\pi)^{2n}, \quad n \in \mathbb{N}. \tag{13}$$

The values of the derivatives at the two ends of the interval are identical, so this is case 1. Therefore, the zeros are

$$f_n T = n, \quad n \in \mathbb{N}$$

if the series converges. With Eq. 13, the series reads

$$\begin{aligned}
\sum_{n=0}^{\infty}\left(\frac{-i}{2\pi fT}\right)^n \frac{d^n\phi}{d(t/T)^n}(1) &= -\frac{1}{2}\sum_{n=1}^{\infty}\left(\frac{-i}{2\pi fT}\right)^{2n}(-1)^n(2\pi)^{2n}\\
&= -\frac{1}{2}\sum_{n=1}^{\infty}\left(\frac{1}{(fT)^2}\right)^n.
\end{aligned}$$

This geometric series converges if

$$fT > 1.$$

Thus the zeros of the Fourier transform are

$$f_n T = 2, 3, 4, \ldots$$

For $fT > 1$ the series converges to

$$\sum_{n=1}^{\infty} \left(\frac{1}{(fT)^2} \right)^n = \frac{1}{1 - \frac{1}{(fT)^2}} - 1 = \frac{1}{(fT)^2 - 1} \, .$$

Thus,

$$X(fT) = \frac{iT}{4\pi} \left(\frac{e^{-2\pi i fT} - 1}{fT\,(1 - (fT)^2)} \right) . \tag{14}$$

In fact, Eq. 14 is also valid when $fT < 1$ and the value at $fT = 1$ exists as a limit. It is identical to the Fourier transform of $x(t)$ obtained by elementary calculations.

Example 2

A function often used in structural dynamics to describe shock loads is

$$\phi\left(\frac{t}{T}\right) = \sin\left(\pi \frac{t}{T}\right) . \tag{15}$$

The deriatives are

$$\begin{aligned}
\frac{d\phi}{d(t/T)} &= \pi \cos\left(\pi \frac{t}{T}\right) \\
\frac{d^2\phi}{d(t/T)^2} &= -\pi^2 \sin\left(\pi \frac{t}{T}\right) \\
\frac{d^3\phi}{d(t/T)^3} &= -\pi^3 \cos\left(\pi \frac{t}{T}\right) \\
\frac{d^4\phi}{d(t/T)^4} &= \pi^4 \sin\left(\pi \frac{t}{T}\right)
\end{aligned}$$

The values at $t = 0$ and $t = T$ are

$$\begin{array}{rclrcl}
\phi(0) &=& 0, & \phi(1) &=& 0 \\
\frac{d\phi}{d(t/T)}(0) &=& \pi, & \frac{d\phi}{d(t/T)}(1) &=& -\pi \\
\frac{d^2\phi}{d(t/T)^2}(0) &=& 0, & \frac{d^2\phi}{d(t/T)^2}(1) &=& 0 \\
\frac{d^3\phi}{d(t/T)^3}(0) &=& -\pi^3, & \frac{d^3\phi}{d(t/T)^3}(1) &=& \pi^3 \\
\frac{d^4\phi}{d(t/T)^4}(0) &=& 0, & \frac{d^4\phi}{d(t/T)^4}(1) &=& 0 \\
& \ldots & & & \ldots &
\end{array}$$

or

$$\frac{d^{2n+1}}{d(t/T)^{2n+1}}(1) = -\frac{d^{2n+1}}{d(t/T)^{2n+1}}(0) = -(-1)^n \pi^{2n+1}, \quad n \in \mathbb{N}_0. \tag{16}$$

Here, the values of the derivatives at the two ends of the interval have opposite sign, so this is case 2. Therefore, the zeros are

$$f_n T = \frac{2n+1}{2}, \quad n \in \mathbb{N}_0 \tag{17}$$

if the series converges. With Eq. 16, the series reads

$$\begin{aligned} \sum_{n=0}^{\infty} \left(\frac{-i}{2\pi f T}\right)^n \frac{d^n \phi}{d(t/T)^n}(1) &= -\sum_{n=0}^{\infty} \left(\frac{-i}{2\pi f T}\right)^{2n+1} (-1)^n \pi^{2n+1} \\ &= \frac{i}{2fT} \sum_{n=0}^{\infty} \left(\frac{1}{(2fT)^2}\right)^n. \end{aligned}$$

This geometric series converges if

$$2fT > 1.$$

Thus the zeros of the Fourier transform are

$$f_n T = \frac{3}{2}, \frac{5}{2}, \frac{7}{2} \ldots$$

For $2fT > 1$ the limit of the series is

$$\sum_{n=0}^{\infty} \left(\frac{1}{(2fT)^2}\right)^n = \frac{1}{1 - \frac{1}{(2fT)^2}} = \frac{(2fT)^2}{(2fT)^2 - 1}$$

entailing

$$X(fT) = \frac{T}{\pi} \left(\frac{e^{-2\pi i f T} + 1}{1 - (2fT)^2}\right). \tag{18}$$

Again, Eq. 14 is also valid when $2fT < 1$ and the value at $2fT = 1$ exists as a limit. Elementary calculation of the Fourier transform also yields Eq. 18.